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This Proposition No. 9 is not true. For when $n = 5$ the number

$$2^{2^n} + 1 = 4{,}294{,}967{,}297 = 641 \cdot 6{,}700{,}417.$$[1]

It is often stated that Fermat did not claim to have proved this theorem. This statement is based on the fact that in a letter to Pascal[2] in August, 1654, he says that, although he is convinced of its truth, he has not been able to find a valid demonstration; and that in a letter to Digby[3] in June, 1658, he says that he is seeking a demonstration of this beautiful theorem. But the quotation given above is from the 1659 letter to Carcavi and in it Fermat seems to say that he has proved it by the method of infinite descent.

For a brief account of Fermat's method of infinite descent, especially of its relation to the solution of Diophantine equations, the reader is referred to R. D. CARMICHAEL'S "*Diophantine Analysis,*" pp. 14–21. The reader is also referred to the supplement of Sir THOMAS L. HEATH'S "*Diophantus of Alexandria,*" Cambridge University Press, 2d edition, 1910, which contains a number of references to the method of infinite descent.

THE EXPONENTIAL AND LOGARITHMIC FUNCTIONS.

By JOSEF NYBERG, Hyde Park High School, Chicago, Ill.

In a previous paper[4] I showed how the study of the line in analytic geometry could be presented from the viewpoint of functions and relations between variables. The present article explains how logarithms and exponential relations are introduced on the same basis.

I begin the work by solving some problem leading to an exponential relation between two variables. Most such problems taken from physics or chemistry lead to a negative exponent, which is undesirable at the beginning of the work. However, a suitable problem is that of determining the number N of bacteria in a solution at a time t, knowing the number N_0 present at $t = 0$ and knowing the rate of increase. The conditions are similar to those for finding the length of a rod with increasing temperature, a problem studied under linear functions, but different in that the coefficient of expansion of the rod is independent of the temperature while the bacteria increase at a rate dependent upon t. If the bacteria increase r per cent. in a unit of time, then

$$N = N_0\left(1 + \frac{r}{100}\right)^t.$$

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[1] This was proved by Euler in 1732. For this and other comments on the theorem see W. W. R. Ball, *Mathematical Recreations and Essays*, 5th edition, p. 39. Also see R. C. Archibald's *Remarks on "Klein's famous problems of elementary geometry,"* this MONTHLY, Vol. 21, p. 248.

[2] *Oeuvres de Fermat*, Vol. 2, p. 307.

[3] *Ibid.*, p. 402.

[4] "The Linear Function and the Line," in this MONTHLY, Nov., 1917, page 406. This was the third of a series of related papers of which the other two are:

"The Unification of Freshman Mathematics," April, 1916, page 101;

"The Presentation of the Notion of Function," Sept., 1917, p. 309.

The next work consists in drawing carefully a graph of $y = a^x$ taking $a = 2$ as the most convenient. We find points on the curve by geometrical constructions using the line $y = ax$ and interpolating points by using the lines $y = \sqrt{a}x$, $y = \sqrt[4]{a}x$, etc. (remembering that the logarithms increase in arithmetic progression when the numbers increase in geometric progression). At once curves like $y = y_0a^x$ can be sketched by a proper multiplication of the ordinates, and $y = y_0a^{nx}$ by a multiplication of the abscissas. Since the latter curve is the same as $y = y_0b^x$ where $b = a^n$, any exponential curve can be sketched from the one fundamental one. Moreover a satisfactory value of n can be found from the graph of $y = a^x$ inasmuch as n is the abscissa of the point whose ordinate is b. We call n the logarithm of b to the base a.

Having seen how the exponential function arises, and how it differs from the linear function in its rate of increase, graphed it, and introduced the word logarithm, we are ready to study the properties of the function. It is at once evident that negative numbers have no real logarithms since the curve by our construction can never get below the x axis, that the logarithm of the base is 1, that $\log 1 = 0$, and that numbers less than 1 have negative logarithms. Progressively we prove the fundamental relations for $\log (N_1/N_2)$, $\log N_1N_2$, and $\log_a N = \log_b N \cdot \log_a b$. For exercises in drill I use numbers whose logarithms can be taken from the graph which the student has drawn, or else use integers less than 40 whose logarithms I have written on the blackboard. The student computes log 3, log .04, etc., from the properties of the function. At this point the student also solves such exponential equations as $p^x \cdot q^{x+2} = r$ $(p, q, r, < 40)$. The difficulties in this equation which Mr. Simpson[1] pointed out can never arise under this method of treatment because negative logarithms are no less common than positive ones, and because the notion of the characteristic and mantissa have not yet been introduced. Mantissas and characteristics are in fact unnecessary until we begin using tables.

Before proceeding to the use of tables I like to explain at this point the significance of the constant e as a base. The fact that $y = e^x$ has at any point a slope equal to the ordinate at that point does not appear to be unusually significant, not even to a student of calculus; nor can the student appreciate its significance as the limiting value of a certain series unless we show him *how* and *why* this particular series is *unavoidable* in the problems. This can be done by returning to the original problem of the bacteria which led to the relation

$$N = N_0\left(1 + \frac{r}{100}\right)^t,$$

when the rate of increase is r per cent. per unit of time. If the unit of time is cut in two, then

$$N = N_0\left(1 + \frac{r}{200}\right)^{2t};$$

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[1] "Relating to the Teaching of Logarithms," in this MONTHLY, Sept., 1916, page 264.

and if divided into q parts, then

$$N = N_0\left(1 + \frac{r}{100q}\right)^{qt}.$$

We wish to derive the expression for N when q is increased indefinitely (*i. e.*, when the unit is decreased in size). Put $r/100q = 1/u$ so that

$$N = N_0\left(1 + \frac{1}{u}\right)^{urt/100}.$$

We may then show that $\lim_{u=\infty}(1 + 1/u)^u$ is a definite constant, which we denote by e. This work may be done with as much rigor as the individual teacher prefers. Thereafter similar problems are assigned from physics, as for example, the amount of light coming through a pane of glass, or problems from chemistry on the decomposition of elements. r is negative for most of these problems, but this fact is not now a disturbing element to the student. These problems also have the advantage that the constants y_0 and n in the relation $y = y_0e^{nx}$ have a definite physical association.

Not until the previous theory is clear do I consider logarithms in their application to computing. There are two new ideas that need to be impressed on the student: (1) the conventions of the computer of dividing a logarithm, irrespective of whether it is positive or negative, into two parts, an integral characteristic and a decimal mantissa; (2) the conventions of the printer used in abbreviating the work of printing the table. Considering also the definition previously given, we are dealing with a logarithm from three distinct aspects: first, what it actually is, as log .6974 = — .15658; second, what the printer puts into the table, as 84342; third, what the computer puts on his paper, as .84342 — 1.

The textbooks do not clearly differentiate these three steps, and yet every error of the beginner is due to the confusion of these notions. It alone is responsible for the error that Mr. Simpson mentions; it leads to the usual errors in handling numbers less than unity, and to errors in finding the antilogarithm. The first point can be emphasized by having the student check his logarithms (if he is working the customary exercises of finding the logarithms of an assigned list of numbers) from his graph. Under the second item I frequently resort to the homely method of making the student put a decimal point before each mantissa in some column of his table. This cures him permanently of any inclination to write .08434 instead of .8434 — 1 for log .6974, or of writing 212.7 instead of 3.2127 for log 1632. The abbreviation of the printer's work is also the justification of 10 as a base in computing, whereas we use e as a base for exponential relations arising in problems other than computing. This the student will readily see if he computes $\log_e 1.234$, $\log_e 12.34$, etc. In fact there is scarcely a better way for learning the rules for handling logarithms than by comparing the similar rules for e as the base. Under the third idea of what the computer puts on his paper, the teacher can explain the advantage of cologarithms, the necessity of writing 2.4133 — 3 instead of .4133 — 1 for log .259 in finding its cube root, and similar matters.

This tripartite analysis of a logarithm is more powerful than may appear at a first reading. We learn by contrast; and the value of this particular explanation lies in seeing logarithms from the viewpoints of the computer, the printer, and the graph. Most teachers are inclined to believe that the best instruction consists in presenting to the student only one idea at a time. Psychologically, however, it is equally sound to keep together two closely related ideas, and teach by contrast and comparisons. Ordinarily logarithms are taught solely for their use in computing; later in analytic geometry the student draws perhaps a few exponential curves; and then in calculus he learns about the base e. The introduction of both bases simultaneously impresses the significant facts about each; the mind is put in a receptive mood when it sees the necessity of the rules. The writer believes the present paper affords a good illustration of how various parts hitherto separated in the curriculum by years can be taught sequentially when organized on the basis of functions. In addition to unity of subject matter, we also obtain an unusual degree of coherence. Each day's work is intimately linked with the preceding and prepares for the following.

NOTE ON LAGRANGE'S LIKE-PRODUCING QUADRINOMIAL.[1]

By Sir THOMAS MUIR, South Africa.

1. From an identity originally due to Lagrange we know that the product

$$(x_1^2 + bcx_2^2 + cax_3^2 + abx_4^2)(y_1^2 + bcy_2^2 + cay_3^2 + aby_4^2)$$

can be expressed in the same form as either factor, namely, in the form

$$P_1^2 + bcP_2^2 + caP_3^2 + abP_4^2.$$

For the case, however, where the second factor is the same as the first, the formula is nugatory, all that it then tells us being that $(x_1^2 + bcx_2^2 + cax_3^2 + abx_4^2)^2$ is equal to itself. It seems therefore not a little interesting to know that without introducing any new principle we can show that the product of the *cube* of the first factor by the second factor can be simply expressed in the particular form in question.

2. We recall in the first place that

$$(x_1^2 + bcx_2^2 + cax_3^2 + abx_4^2)^2 = \begin{vmatrix} x_1 & bcx_2 & cax_3 & abx_4 \\ -x_2 & x_1 & -ax_4 & ax_3 \\ -x_3 & bx_4 & x_1 & -bx_2 \\ -x_4 & -cx_3 & cx_2 & x_1 \end{vmatrix}$$

and that each primary minor of this determinant contains the square root of the

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[1] Part of the object of this paper is to throw a little light on the subject of Problem 489 of this MONTHLY for Nov. 1917.